1. The complex number $i$ is defined by the fact that $i^2 = -1$; note that as soon as we have $i$, we also have the number $-i$, which has exactly the same property!

$(-i)^2 = i^2 = -1$, too!

2. (a) Cartesian form of a complex number is the form $a + bi$, where $a, b \in \mathbb{R}$. (i.e., $a$ and $b$ are real #'s)

(b) The real part of $a + bi$ is $a$; the imaginary part of $a + bi$ is $b$.

(c) We denote the set of all complex numbers by "$\mathbb{C}$"; if we consider Cartesian form of a complex $a + bi$ to correspond to the point $(a, b)$ in the Cartesian plane, we can visualize $\mathbb{C}$ as the "complex plane", containing the real line as its x-axis:

3. The conjugate of a complex number $a + bi$ is $\overline{a + bi} = a - bi$ (just negate the imaginary part)

The basic properties of the operation of conjugation are as follows:

- $\overline{\overline{z}} = z$ (i.e., conjugation is an "involution")
- If $z$ is real, $\overline{z} = z$
- $\overline{z - w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z} \overline{w}$
- $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$

4. (a) We can add + subtract complex #'s in Cartesian form by simply treating $i$ as a variable:

$$(1 + i) + (2 - 2i) = 3 - i; \quad (1 + i) - (2 - 2i) = -1 + 3i$$

(b) Using the fact that $i^2 = -1$, we can easily multiply complex #'s in Cartesian form simply by distributing:

$$(1 + i)(2 - 2i) = 2 + 2i - 2i - 2i^2 = 2 - 2(-1) = 2 + 2 = 4$$

(c) We can compute the quotient of two complex #'s in Cartesian form via the trick of multiplying top + bottom by the conjugate of the denominator and simplifying:

$$\frac{1+i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{(1+i)(2+3i)}{(2-3i)(2+3i)} = \frac{-1 + 5i}{2^2 - (3i)^2} = \frac{-1 + 5i}{4 + 9}$$

6. The "<" operation doesn't work for complex #'s: consider $i$; since $i > 0$, either $i > 0$ or $-i > 0$.

In the first case, since $i > 0$ and $i > 0$, $2i > 0, \overline{2i} > 0 (x)$

In the second case, since $-i > 0$ and $-i > 0$, $(-i)(2i) > 0, \overline{(-i)(2i)} > 0 (x)$

---i.e., there's no way of consistently declaring $i$ to be positive or negative!
6. (a) \((2+i) + (4-3i) = \frac{6-2i}{2}\)  
(b) \((2+i) - (4-3i) = \frac{-2+4i}{2}\)  
(c) \((2+i)(4-3i) = 8+4i-6i-3i^2 = \frac{11-2i}{2}\)  
(d) \(\frac{2+i}{4-3i} \cdot \frac{4+3i}{4+3i} = \frac{8+6i+6i+3i^2}{45} = \frac{5+10i}{25} = \frac{1}{5} + \frac{2}{5}i\)  
(e) \((3+2i) + \frac{3+2i}{3+2i} = (3+2i) + (3-2i) = 6\)  
(f) \((3+2i) - \frac{3+2i}{3+2i} = (3+2i) - (3-2i) = 4i\)  
(g) \((3+2i)(3+2i) = (3+2i)(3-2i) = 9+4 = \frac{13}{2}\)  
(h) \(\frac{1}{2i} \cdot \frac{-2i}{-2i} = \frac{-2i}{2} = \frac{-1}{2}i\)  

7. (a) \(2^0 = 1\);  
\(2^1 = 2\);  
\(2^3 = 4\);  
\(2^5 = 16\)  
(b) \(i^0 = 1\);  
\(i^1 = i\);  
\(i^3 = -i\);  
\(i^4 = 1\)  
(c) \((1+i)^0 = 1\);  
\((1+i)^1 = 1+i\);  
\((1+i)^3 = (1+i)(1+i) = 2i\);  
\((1+i)^5 = 2i(1+i) = -2+2i\);  
\((1+i)^7 = (-2+2i)(1+i) = -4i\);  
\((1+i)^9 = -4i(1+i) = -8i\);  
\((1+i)^8 = (8-8i)(1+i) = 16\).
8. (a) Let \( z = a + bi \), where \( a, b \in \mathbb{R} \).

If \( \bar{z} = z \), then \( a - bi = a + bi \),
\[
\begin{align*}
0 &= 2bi \\
\therefore b &= 0, \text{i.e., } z \text{ is a real number}.
\end{align*}
\]

(b) Let \( z = a + bi \), and write \( z = a + bi \), where \( a, b \in \mathbb{R} \).

Then \( \overline{z^2} = (a + bi)(a + bi) = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2 \).
\( a^2 + b^2 \), which is real and \( \geq 0 \) since \( a, b \in \mathbb{R} \).

(c) Let \( z = a + bi \), where \( a, b \in \mathbb{R} \).

Then \( \overline{z} = a - bi \).

Adding these equations,
\[
2a = 2a, \text{ so } a = \frac{1}{2}(x + \bar{x}).
\]
I.e., The real part of \( z \) is \( \frac{1}{2}(x + \bar{x}) \).

Subtracting them,
\[
2b = 2bi, \text{ so } b = \frac{1}{2}i(\bar{x} - x).
\]
I.e., The imaginary part of \( z \) is \( \frac{1}{2}i(\bar{x} - x) \).

9. (This is a lot easier than it might at first look — just write down the hypothesis, conjugate both sides, and carefully simplify!)

Suppose that \( P(z) = a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \ldots + a_{n-1} \overline{z}^{n-1} + a_n \overline{z}^n \),
where \( a_0, a_1, \ldots, a_n \in \mathbb{R} \).

If \( P(z) = 0 \), then \( a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \ldots + a_{n-1} \overline{z}^{n-1} + a_n \overline{z}^n = 0 \)
conjugating both sides,
\[
\overline{a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \ldots + a_{n-1} \overline{z}^{n-1} + a_n \overline{z}^n} = 0
\]
so \( \overline{a_0} + \overline{a_1 \overline{z}} + \overline{a_2 \overline{z}^2} + \ldots + \overline{a_{n-1} \overline{z}^{n-1}} + \overline{a_n \overline{z}^n} = 0 \).

Thus \( \overline{a_0} + \overline{a_1 \overline{z}} + \overline{a_2 \overline{z}^2} + \ldots + \overline{a_{n-1} \overline{z}^{n-1}} + \overline{a_n \overline{z}^n} = 0 \).

Therefore \( \overline{a_0} + \overline{a_1 \overline{z}} + \overline{a_2 \overline{z}^2} + \ldots + \overline{a_{n-1} \overline{z}^{n-1}} + \overline{a_n \overline{z}^n} = 0 \).

But all coefficients \( a_0, a_1, a_2, \ldots, a_n \in \mathbb{R} \), so
\[
a_0 + a_1 \overline{z} + a_2 \overline{z}^2 + \ldots + a_{n-1} \overline{z}^{n-1} + a_n \overline{z}^n = 0
\]
I.e., \( P(z) = 0 \).