Remarks

At last we arrive at the last full chapter we’ll cover! With Chapter 3 behind us, we now prove the full results we’ve been building toward regarding holomorphic functions on convex simply-connected open sets, integrals around triangles arbitrary cycles, and power series in disks Laurent series in annuli.

A focal question has been, and will continue to be:

For \( f \in H(V) \) and \( C^* \subset V \), under what conditions can we prove that \( \oint_C f(z) \, dz = 0 \)?

Several results we’ve proven, and several we’ll prove in Chapter 4, give some sort of condition such that either force this integral to be zero for all \( f \) (e.g., if \( C \) is the boundary of a closed triangle in \( V \)) or force this integral to be zero for all \( C \) (e.g., if \( f \) has an antiderivative). Our one big example of a nice function not integrating to zero along a closed curve was the function \( f(z) = \frac{1}{z} \) integrated around the unit circle—in this chapter, we’ll finally sort that out and see that, in essence, this is the model for the only phenomenon that ever causes such integrals to be nonzero!

In essence, the first segment of the chapter takes a very close look at arguments and polar form in \( \mathbb{C} \), which are intimately related to the complex logarithm, in order to rigorously define what we mean by how many times a path goes around some point in the complex plane. For clarity, Ullrich uses “log” to mean the complex [multiple-valued] logarithm and “\( \ln \)” for the old fashioned real-valued natural logarithm on \((0, \infty)\). The material at the start of this chapter will look very technical at first, but with the right pictures and thoughts, we’ll find that what looks like a mess of symbols is actually quite intelligible.

First, a result extending Prop. 2.3 that we really could have proven back then (but we proved just what we needed to at the time):

**Proposition 4.0:** If \( f \) is continuous on a convex connected open set \( V \) and \( \int_{\gamma} f(z) \, dz = 0 \) for any triangle \( \mathcal{T} \subset V \) piecewise-smooth closed path \( \gamma \) in \( V \), then \( \exists F : V \to \mathbb{C} \) with \( F' = f \).

- The proof of this parallels that of Prop. 2.3: choose \( z_0 \in V \) and for each \( z \in V \) define \( F(z) = \int_{\gamma} f(z) \, dz \), where \( \gamma \) is any path from \( z_0 \) to \( z \). That this doesn’t depend on the choice of path \( \gamma \) is a consequence of the hypothesis on closed paths, and that \( F' = f \) works just the same as in Prop. 2.3 (except that the closed path involved isn’t a triangle anymore).

**A branch of the logarithm** in an open set \( V \subset \mathbb{C} \) is a function \( L \in H(V) \) such that \( \forall z \in V, \, e^{L(z)} = z \), i.e., an inverse on \( V \) to the exponential function.

- It’s actually sufficient for \( f \) to be merely continuous—holomorphicity follows from the fact it inverts the exponential function. [Prop. 4.1]; in fact, differentiating the given equation shows that \( L'(z) = \frac{1}{z} \).
- If \( V \) is connected, this is equivalent to having \( L \) having \( L'(z) = \frac{1}{z} \) and there being some \( z_0 \in V \) for which \( e^{L(z_0)} = z_0 \), i.e., having the right derivative and inverting the exponential function at just one point. [Prop. 4.2]
- Thus there exists a branch of the logarithm on \( V \) just when there exists a function \( f \in H(V) \) with \( f'(z) = \frac{1}{z} \) [Prop. 4.3]; from Chapter 3, we know this is true for convex \( V \). [Prop. 4.4]
- Our final preliminary observation is that this is equivalent to the statement that \( \oint_C \frac{1}{z} \, dz = 0 \) for all smooth closed curves \( \gamma \) in \( V \). [Prop 4.5], which returns us back to our old friend!
Remarks, cont’d

- The **index** of a path $\gamma$ about 0, $\text{Ind}(\gamma, 0)$, essentially counts how many times $\gamma$ wraps counterclockwise around 0. We’ll end up with a nice integral formula for it, but it’s wrong-minded to define it that way. **Lemma 4.6** paves the way for us to define $\text{Ind}(\gamma, 0)$ properly for a continuous path $\gamma : [0, 1] \to C \setminus \{0\}$; we essentially write $\gamma$ in polar form and focus on the angle, ignoring the radius. (Remember that the transformation $z \mapsto \frac{z}{|z|}$ simply maps each nonzero complex number to the unit circle by dividing it by its length.)

  (i) If we choose a starting argument for $\gamma(0)$, we can continue that choice along $\gamma$ continuously in a unique way: if $\theta_0 \in \mathbb{R}$ has $e^{i\theta_0} = \frac{\gamma(0)}{|\gamma(0)|}$, then there exists a unique continuous function $\theta : [0, 1] \to \mathbb{R}$ such that $\theta(0) = \theta_0$ and $\gamma(t) = |\gamma(t)| e^{i\theta(t)}$.

  (ii) Nudging the path a little just nudges the angles a little: $\forall \varepsilon > 0$, $\exists \delta > 0$ so that if $\tilde{\gamma} : [0, 1] \to \mathbb{C}$ is continuous and satisfies $|\tilde{\gamma}(t) - \gamma(t)| < \delta$ for $t \in [0, 1]$, then there is a continuous function $\tilde{\theta} : [0, 1] \to \mathbb{R}$ such that for all $t \in [0, 1]$, $\tilde{\gamma}(t) = |\tilde{\gamma}(t)| e^{i\tilde{\theta}(t)}$ and $|\tilde{\theta}(t) - \theta(t)| < \varepsilon$.

  (iii) If $\gamma$ is a closed curve, then $\frac{1}{2\pi i} [\theta(1) - \theta(0)] \in \mathbb{Z}$;
   if $\gamma$ is a smooth closed curve, then this value is given by $\frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} \, dz$, which brings us back again to our old friend. We could have defined the index this way, and, indeed, some sources do; but this involves integrals and only works for nice curves—our definition only requires continuity.

In general, given $a \in \mathbb{C}$ and a continuous closed curve $\gamma : [0, 1] \to \mathbb{C} \setminus \{a\}$:

- Apply the above result to result to $\gamma(t) - a$ to get $\gamma(t) = a + r(t) e^{i\theta(t)}$, i.e., write $\gamma(t)$ in polar form centered at $a$.

- Define the **index** of $\gamma$ about $a$ as $\text{Ind}(\gamma, a) = \frac{1}{2\pi} [\theta(1) - \theta(0)] \in \mathbb{Z}$, i.e., how much the angle has changed as we moved along $\gamma$ (divided by $2\pi$ so as to count a full circuit as 1).

- If $\gamma$ is smooth, $\text{Ind}(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - a} \, dz$.

- For a given cycle $\Gamma$, $\text{Ind}(\gamma, z)$ is an integer-valued function for $z \in \mathbb{C} \setminus \Gamma^*$ that’s constant on each component of that set (and zero on the unbounded component). [Prop. 4.7]

- Finally, we can re-state Prop. 4.5 in terms of index: there exists a branch of the logarithm on an open set $V \subset \mathbb{C} \setminus \{0\}$ just when $\text{Ind}(\gamma, 0) = 0$ for all closed curves $\gamma$ in $V$. [Prop. 4.5.1]

**Problem assignments**

None yet, but we’ll do some integral computations once we have enough material under our belts!