Remarks [throughout, \( V \) represents an open set in \( \mathbb{C} \)]

In the second half of Chapter 3 (which functions largely as an independent chapter or two), we analyze [isolated] zeros and singularities of analytic functions. Zeros come first: if an analytic function \( f \) has \( f(z_0) = 0 \), then either there exists a smallest \( N \) for which \( f^{(n)}(z_0) \neq 0 \), or else all derivatives of \( f \) vanish at \( z_0 \) and \( f \) is identically zero. In the former case we say \( f \) has a **zero of order** \( N \) at \( z_0 \) and we can factor out \((z - z_0)^N\) from \( f \):

- **Lemma 3.9:** If \( f \in H(V) \) has a zero of order \( N \) at \( z_0 \), then there exists some \( g \in H(V) \) with \( g(z_0) \neq 0 \) and \( f(z) = (z - z_0)^N g(z) \).
  - From this, we can immediately conclude that any zero \( z_0 \) of a nonconstant analytic function must be **isolated**, i.e., there must exist some \( r > 0 \) such that \( f \) has no other zeros in \( D(z_0, r) \).
  - **Corollary 3.10:** If \( V \) is connected and \( f \in H(V) \) has a **non-isolated** zero, then \( f \) must be identically zero.

This also allows us to prove the **Maximum Modulus Theorem:** We’ll look only at the “first proof” given, which uses our analysis of zeros of an analytic function; skip Proposition 3.12 through the second proof.

- **Theorem 3.11:** If \( V \) is connected and \( f \in H(V) \), then \( |f| \) cannot achieve a **local maximum** on \( V \).

What was a provable fact (that zeros of a [nonconstant] analytic function are isolated) isn’t automatic for singularities, i.e., points at which a function is **not** analytic, so we make some definitions:

- The **punctured disk** centered at \( z_0 \) with radius \( r \) is \( D'(z_0, r) = \{ z \in \mathbb{C} : 0 < |z - z_0| < r \} \).
- \( f \in H(V) \) has an **isolated singularity** at \( z_0 \notin V \) if \( D'(z_0, r) \subset V \) for some \( r > 0 \).
  - i.e., \( f \) isn’t necessarily even defined at \( z_0 \), but it is both defined and **analytic** on some punctured disk around \( z_0 \).

We classify isolated singularities of \( f \) as one of three types: removable singularities, poles, and essential singularities—the remainder of the chapter establishes basic results about these types of singularities. Suppose that \( f \) has an isolated singularity at \( z_0 \). Then:

- \( f \) has a **removable singularity** at \( z_0 \) if we can define \( f(z_0) = a \) for some \( a \in \mathbb{C} \) so that \( f \) is analytic on all of \( D(z_0, r) \) for some \( r > 0 \).
  - This condition is equivalent to \( f \) being **bounded** on a neighborhood of \( z_0 \). \[\text{Lemma 3.13}\]

- \( f \) has a **pole** at \( z_0 \) if \( \lim_{z \to z_0} f(z) = \infty \).
  - This condition is equivalent to \( f \) being representable as a power series with some finite number of negatively-powered terms: there exist \( N \geq 1 \) and coefficients \( c_n \in \mathbb{C} \) for \( n \geq -N \) such that \( f(z) = \sum_{n=-N}^{\infty} c_n(z - z_0)^n \) on some punctured disk around \( z_0 \). \[\text{Proposition 3.14(i)}\]
  - We call the part with just the negatively-powered terms, i.e., \( \sum_{n=-N}^{-1} c_n(z - z_0)^n \), the **principal part** of \( f \) near \( z_0 \)—the principal part contains all of the “pole” bits of the function, i.e., \( f \) minus its principal part has just a removable singularity at \( z_0 \).

- In any other case, \( f \) has an **essential singularity** at \( z_0 \), which is equivalent to the statement that for all \( c \in \mathbb{C} \), and any \( \varepsilon > 0 \), there exist points \( z \in D'(z_0, r) \) for which \( f(z) \in D(c, \varepsilon) \). \[\text{Proposition 3.14(ii)}\]
Remarks (continued)

This all comes together nicely from the perspective of *Laurent series*:

- If $f(z) = \sum_{n=-\infty}^{\infty} c_n(z - z_0)^n$ is analytic on $D'(z_0, r)$, then:
  - $f$ has a removable singularity at $z_0$ just when $c_n = 0$ for all $n < 0$
    (i.e., the principal part of $f$ is zero).
  - $f$ has a pole of order $N$ at $z_0$ just when $c_{-N} \neq 0$ and $c_n = 0$ for $n < -N$
    (i.e., the negative powers terminate, and the principal part of $f$ is a polynomial in $\frac{1}{z - z_0}$).
  - $f$ has an essential singularity at $z_0$ just when the negative powers don’t terminate.

**Problem assignments**: 3.6, 3.7, 3.9, 3.11, 3.19, 3.20