So as not to stray too far into the realm of metric topology, please skip the segment starting with “We can use…” at the bottom of p. 35 and pick up again at “Liouville’s Theorem…” at the top of p. 37.

Remarks
With “preliminary” results established, we’re now getting into the meat of the material with “elementary” results—by the end of this chapter, we’ll know enough to eventually do some actual problems! Note that the order of this discussion differs from that of the text (so you’ll have two paths along this trail).

We already know that a power series that vanishes on a small disk around its center point must be identically zero; we can actually do a fair bit better:

- **Corollary 3.8**: Suppose that $V \subset \mathbb{C}$ is open and connected and that $f \in H(V)$.
  
  If all derivatives of $f$ vanish at some point $z_0 \in V$, then $f$ is constant on $V$.

The CIF disks allowed us to evaluate a holomorphic function $f$ at a point inside a disk $D$ via its values on $\partial D$; the **Cauchy Integral Formula for derivatives** (disk version) gives us the same for $f$’s derivatives, following the method of the earlier CIF and our formula for $f^{(n)}(z_0)$ in Chapter 2:

- **Corollary 3.1**: Suppose that $V \subset \mathbb{C}$ is open and $f \in H(V)$. If $D$ is any disk with $\bar{D} \subset V$, then:
  
  $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{n+1}} \, dw$$
  
  for all $z \in D$.

**Cauchy’s Estimates** simply apply the ML Inequality to the integral above (at $z_0$):

- **Corollary 3.2**: Suppose that $V \subset \mathbb{C}$ is open and $D$ is a disk centered at $z_0$ with $\bar{D} \subset V$.
  
  If $|f| \leq M$ on $\bar{D}$, then $|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}$ for $n = 0, 1, 2, \ldots$

This simple estimate has surprisingly strong consequences:

- **Liouville’s Theorem**: If $f$ is entire* and bounded, then $f$ is constant.
- **Fundamental Theorem of Algebra**: Every polynomial of positive degree has at least one root.
  
  - Note that the polynomial is a complex polynomial, and the guaranteed root could be complex (even if the polynomial has all real coefficients).

Two results tell us that uniform convergence buys us a lot more for holomorphic functions than it did for real functions.

- If $V \subset \mathbb{C}$ is open, $f_n \in H(V)$ for all $n$, and $(f_n) \to f$ uniformly on compact subsets of $V$, then:
  
  - **Corollary 3.0**: $f \in H(V)$, and
  
  - **Proposition 3.5**: $(f'_n) \to f'$ uniformly on compact subsets of $V$.

Note the use of “uniformly on compact sets” in these results—we don’t need or expect uniform convergence on the entire open set $V$ (think of the disk of convergence for a power series), but rather on compact subsets of $V$.

* i.e., holomorphic on all of $\mathbb{C}$

**Problem assignments**: 3.1, 3.4, 3.5

For 3.4 and 3.5, work to unravel the statement into something that you can attack with Liouville’s Theorem [3.4] or its proof [3.5].