Remarks

Finally, we arrive at the promised result connecting complex-differentiable functions and power series, which follows almost immediately from the Cauchy Integral Formula for disks:

**Theorem 2.6:** If $f$ is differentiable on an open set $V$, and if $D$ is a disk centered at $z_0$ with $\bar{D} \subset V$,

then $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ for all $z \in D$, where $c_n = \frac{1}{2\pi i} \oint_{\partial D} \frac{f(w)}{(w - z_0)^{n+1}} \, dw$.

This tells us a great many things:

- Directly, $f$ being complex-differentiable on an open set means that $f$ is represented by a power series near each point of $V$.

- Consequently, $f$ shares all *local properties* of power series near each of its points.

- In particular, $f$ must be differentiable infinitely many times at $z_0$; specifically, under the hypotheses of this theorem, we have $f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\partial D} \frac{f(w)}{(w - z_0)^{n+1}} \, dw$ for all $z \in D$.

The proof of this result hinges on a simple observation: the factor $\frac{1}{w - z}$ in the Cauchy Integral Formula can be written as the sum of a particular geometric series:

$$\frac{1}{(w - z_0) \left( \frac{z-z_0}{w-z_0} \right)} = \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left( \frac{z-z_0}{w-z_0} \right)^n.$$

Note that this geometric series converges for $\left| \frac{z-z_0}{w-z_0} \right| < 1$, i.e., when $|z-z_0| < |w-z_0|$, which is the radius of the disk $D$ (thus we have absolute convergence on the interior of $D$ and uniform convergence on any smaller disk around $z_0$). Plugging this into the CIF and rearranging terms, the power series falls right into our lap.

Finally, the related definitions of a function being *analytic* or *holomorphic* on a set $V$ are discussed, along with the notation “$H(V)$” for the set of all functions holomorphic on $V$.

**Food for thought**

Give Chapter 1 a fresh look, knowing that everything we proved about power series will actually be true locally for any holomorphic function!