Upper bounds and the supremum  Suppose that $A \subset \mathbb{R}$ is a set of real numbers.

- $M$ is an upper bound for $A$ means that for all $a \in A$, $a \leq M$.

- $s$ is the supremum (least upper bound) of $A$ means that:
  - (i) $s$ is an upper bound for $A$; and
  - (ii) if $M$ is an upper bound for $A$, then $s \leq M$.

  - If we declare that $-\infty < x < \infty$ for every real number $x$, then every set in $\mathbb{R}$ has a supremum.

Sequences  A sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ is an infinite ordered list $a_1, a_2, a_3, \ldots$ of real numbers (its terms).

- $\lim_{n \to \infty} a_n = L$ means: for all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ so that if $n \geq N$, then $|a_n - L| < \varepsilon$.

  - This means that if we toss out more and more initial terms of the sequence, the rest eventually squeeze as close to $L$ as we like.

  - A sequence doesn’t necessarily have a limit; but if it does, that limit is unique.

Series  Recall that the value of a series $\sum_{n=n_0}^{\infty} a_n$ is defined as $\lim_{N \to \infty} \left[ \sum_{n=n_0}^{N} a_n \right]$ (when that limit exists!).

- If this limit of partial sums exists, we call the series convergent; otherwise, we call it divergent.

- $\sum_{n=n_0}^{\infty} a_n$ is absolutely convergent means that $\sum_{n=n_0}^{\infty} |a_n|$ converges.

  - in this case, the series itself is convergent, and its sum is the same no matter how we rearrange its terms.

- If $\sum_{n=n_0}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$ (but not vice-versa!).

  - Consequence: if $\lim_{n \to \infty} a_n \neq 0$, then $\sum_{n=n_0}^{\infty} a_n$ diverges (but also not vice-versa!).
Limits and continuity Let $f$ be a real-valued function

- \( \lim_{x \to a} f(x) = L \) means that: for all \( \varepsilon > 0 \), there exists some \( \delta > 0 \) so that if \( 0 < |x - a| < \delta \), then \( |f(x) - L| < \varepsilon \).
  - This means that as we look only at values of \( x \) closer and closer to (but not equal to) \( a \), the values \( f(x) \) squeeze as close as we like to \( L \).

- \( f \) is continuous at \( a \) means that \( \lim_{x \to a} f(x) = f(a) \), i.e., for all \( \varepsilon > 0 \), there exists some \( \delta > 0 \) so that if \( |x - a| < \delta \), then \( |f(x) - f(a)| < \varepsilon \).
  - This means that as we look only at values of \( x \) closer and closer to \( a \), the values \( f(x) \) squeeze as close as we like to \( f(a) \).

- \( f \) is continuous means that \( f \) is continuous at each point of its domain.

Sequences and series Let \((f_n)_{n \in \mathbb{N}}\) be sequence of real-valued functions having the same domain \( A \).

- \((f_n)\) converges to a function \( f \) pointwise means that for each \( x \in A \), \( \lim_{n \to \infty} f_n(x) = f(x) \), i.e., for all \( x \in A \) and for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) so that if \( n \geq N \), then \( |f_n(x) - f(x)| < \varepsilon \).
  - Note that for a given target distance \( \varepsilon \), the values of \( N \) needed at different points \( x \) can vary, so the rate of squeezing to \( f \) can vary from point to point along the domain. For this reason, concepts (limits, continuity, integrals, etc.) that depend on more than just the value of a function at one point don’t necessarily behave well when we just have pointwise convergence.

- \((f_n)\) converges to a function \( f \) uniformly means that: for all \( \varepsilon > 0 \), there exists \( N \in \mathbb{N} \) so that for all \( x \in A \), if \( n \geq N \), then \( |f_n(x) - f(x)| < \varepsilon \).
  - Due to the reordering of quantifiers, for each target distance \( \varepsilon \), a single value of \( N \) will work for all points \( x \) in the domain simultaneously. As a result, this stronger sense of convergence lets us draw (and prove) conclusions about the limits, continuity, and integrals for the limit function \( f \) in terms of the corresponding properties of the functions in the sequence \((f_n)\).

- A series \( \sum_{n=n_0}^{\infty} f_n \) of functions is defined via partial sums, just as with series of real numbers.

The Weierstrass M-Test is a very useful tool for determining uniform convergence of series (and, in fact, sequences!) of functions:

If \( M = \sum_{n=n_0}^{\infty} M_n \) is a convergent series of nonnegative numbers,

and if \( |f_n(x)| \leq M_n \) for each \( x \in A \) and \( n \geq n_0 \),

then \( \sum_{n=n_0}^{\infty} |f_n| \) converges uniformly to some function \( f \), and \( f(x) \leq M \) for all \( x \in A \).

- Note that this allows us to conclude that \( \sum_{n=n_0}^{\infty} f_n \) is uniformly convergent to some function \( f \) with \( |f(x)| \leq M \) for all \( x \in A \).